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Long Paths and Cycles in Dynamical Graphs

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We study the large-time dynamics of a Markov process whose states are finite directed graphs. The number of the vertices is described by a supercritical branching process, and the edges follow a certain mean-field dynamics determined by the rates of appending and deleting. We find sufficient conditions under which asymptotically *a.s.* the order of the largest component is proportional to the order of the graph. A lower bound for the length of the longest directed path in the graph is provided as well. We derive an explicit formula for the limit as time goes to infinity, of the expected number of cycles of a given finite length. Finally, we study the phase diagram.

KEY WORDS: Branching processes; dynamical random graphs; phase transition; randomly grown networks.

1. INTRODUCTION

Dynamical random networks has become a subject of intensive study in mathematics as well as in physics over the last few years (see, e.g., a review by Strogatz[(1)]). Here are some examples: random grammars on graphs (Malyshev[(2)]), the "small-world" networks (Watts and Strogatz,[(3)] Barbour and Reinert[(4)]), scale-free networks (Barabási *et al.*,[(5)] Bollobás *et al.*[(6)]), randomly grown networks (Callaway *et al.*[(7)]). We shall study a random graph grammar model that shares properties of branching processes, random graphs and randomly grown networks affected by the aging of the connections as well as nodes.

Starting with the paper by Erdös,[(8)] the central and most studied model in random graph theory is $G_{n,p}$, which is a graph on n vertices with a probability p of any edge. The most intriguing feature of this model is the

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phase transition at $p = 1/n$. For detailed studies of this model consult Bollobás,[9] and Janson *et al.*[10] The graph process we consider here can be viewed as a dynamic non-homogeneous generalization of the model $G_{n,p}$, with n being replaced by a random process with non-decreasing in time unbounded trajectories and with the probability of an edge between any two vertices being a function of time and of the vertices themselves. More precisely, our model is defined as follows.

Consider on some probability space $(\Omega, \Sigma, \mathbf{P})$ a Markov process with the states in the space of finite graphs with directed multiple edges (i.e., zero, one, or even more than one edge are allowed in either direction between two nodes); in short, directed multi-graphs. The evolution of this process is described by the rates of appending new vertices and edges, and the rate of deleting edges as follows. Let $V(t)$ denote the set of vertices at time t. We shall define two types of edges, call them type 1 and type 2 and denote their sets $\mathscr{L}^1(t)$ and $\mathscr{L}(t)$, respectively, at time t. Assume, at $t = 0$ there is one vertex with no edges, i.e., we set $|V(0)| = 1$ and $\mathscr{L}^1(0) = \mathscr{L}(0) = \varnothing$. Then from each vertex of the graph we draw with rate λ_1 a new type 1 edge to a new vertex. In another words with every vertex in the graph we associate a Poisson process with intensity λ_1, every occurrence of which corresponds to the appearance of a new type 1 edge and a new vertex in the graph. As soon as there are at least two vertices in the graph, from each vertex we draw with rate λ_2 a new type 2 edge to a vertex, which we choose with equal probabilities among the rest of existing vertices in the graph. By doing this we add a new edge to the set $\mathscr{L}$. Any edge in the graph is deleted with rate μ, which means that the lifetime of any edge is exponentially distributed with mean value $1/\mu$. Thus we shall call the age of an edge the time since its appearance. Finally, we assume that all the processes of appending and deleting are independent. Malyshev[2] first introduced and studied this process.

Notice that the subgraph associated with the type 2 edges of this model interpolates between two well-studied classes of models, which have widely different properties. If we do not delete the edges in our model, i.e., let $\mu = 0$, we obtain a continuous-time generalization of the randomly grown network. On the other hand, when we let $\mu \to \infty$, our graph behaves like the $G_{n,p}$ model (for the details we refer to Turova[11]). Hence, this model provides a unified overview of two very different models and the relations between them. In particular, it allows one to explain the dramatic difference in the macro-behaviour of randomly grown networks and random graphs as observed previously by physicists (see Callaway *et al.*[7]).

Recall that the processes studied in random graph theory are defined on a fixed set of vertices, while the graph acquires more edges. In our model the number of vertices $|V(t)|$ is a random process itself, it is a binary fusion (or Yule process) having $\mathbf{E}\,|V(t)| = e^{\lambda_1 t}$ and the asymptotic behaviour

$$|V(t)|\, e^{-\lambda_1 t} \Rightarrow \xi \qquad \text{as} \quad t \to \infty,$$

where ξ follows Exp(1)-distribution (see, e.g., Athreya and Ney[12]). The dynamics of this process breaks the homogeneity of the graph structure. Furthermore, by deleting edges in our model we lose another nice property of the graph process associated with $G_{n,p}$, namely, the monotonicity of acquiring edges. All this makes the standard results in random graph theory not readily applicable.

Besides its novelty and close relation to areas of current interest in probability theory, discrete mathematics and physics, another strong reason to investigate this type of process is its potential application in social sciences and biology. Indeed, our model captures the following properties of the social networks: the uniform boundedness of the degrees of the vertices and the decay of the old connections (Jin *et al.*[13]).

Neural networks provide especially nice examples of complex dynamic random graph structure (see, e.g., Xing and Gerstein[14]). In this setup the vertices of the graph denote neurons, $V(t)$ represents an area of current interaction, while the appearance of any new edge is treated as an impulse between the corresponding neurons along synaptic connections. Naturally, the life-time of an edge represents the duration of an impulse. Thus we get a model for the spreading of the impulses in a neural network, since the spike trains of the neurons are often approximated by a Poisson process (see, e.g., Prut *et al.*[15]).

Just as the theory of random graphs is mainly concerned with asymptotic behavior, our main interest here in our study, as far as applications, is in the large-time properties of our dynamic graph model. We shall make further steps towards the complete description of the phase diagram for our model. We note that Malyshev[2] obtained the first results on this model.

The Main Results

We investigate the dynamics of the directed multi-graph defined above on the set of vertices $V(t)$, and whose set of edges is $\mathscr{L}(t)$, i.e., the type 2 edges only. We shall denote this graph

$$\mathscr{G}^{d,m}(t) = (V(t), \mathscr{L}(t)), \qquad t \geqslant 0,$$

where index "d" stands for "directed" and "m" stands for "multi-." For any directed multi-graph $\mathscr{G}^{d,m}$ we shall define a directed graph $\mathscr{G}^{d}$ and a non-directed graph $\mathscr{G}$ in the following natural way. Either of these graphs has the same set of vertices as $\mathscr{G}^{d,m}$. There is an edge from vertex v to vertex v' in $\mathscr{G}^{d}$ if and only if there is at least one edge from vertex v to vertex v'

in $\mathscr{G}^{d,m}$. Correspondingly, there is a non-directed edge between two vertices in $\mathscr{G}$ if and only if there is at least one edge between these two vertices in $\mathscr{G}^{d,m}$. An ordered set of different vertices in a directed graph is called a directed path, if from any vertex of this set except the last one, there is an edge to the consecutive vertex of this set. The length of the path is the number of its edges.

To formulate our results we introduce first for any $T>0$, $\lambda_1>0$ and $\mu \geqslant 0$, a function

$$\theta(T, \lambda_1, \mu) = \begin{cases} \dfrac{e^{-\mu T} - e^{-\lambda_1 T}}{\lambda_1 - \mu}, & \text{if} \quad \mu \neq \lambda_1, \\ e^{-\lambda_1 T}\, T, & \text{if} \quad \mu = \lambda_1, \end{cases} \tag{1.1}$$

and derive

$$\bar{\theta}(\lambda_1, \mu) \equiv \max_{T>0} \theta(T, \lambda_1, \mu) = \begin{cases} \dfrac{1}{\lambda_1} \left(\dfrac{\mu}{\lambda_1} \right)^{\frac{\mu/\lambda_1}{1-\mu/\lambda_1}}, & \text{if} \quad \mu \neq \lambda_1, \\ \dfrac{1}{\lambda_1}\, e^{-1}, & \text{if} \quad \mu = \lambda_1. \end{cases} \tag{1.2}$$

Theorem 1.1. (I) If the parameters $\lambda_1, \lambda_2, \mu$ satisfy

$$2\lambda_2 \bar{\theta}(\lambda_1, \mu) > 1 \tag{1.3}$$

then with probability tending to one as $t \to \infty$, the largest connected component of the graph $\mathscr{G}(t)$ contains at least $\alpha\, |V(t)|$ vertices, where

$$\alpha = \sup_{T>0 \,:\, 2\lambda_2 \theta(T, \lambda_1, \mu) > 1} \beta(T)\, e^{-\lambda_1 T}$$

with $\beta(T) = \beta(T, \lambda_1, \lambda_2, \mu)$ defined by

$$\beta(T) + \exp\{-\beta(T)\, 2\lambda_2 \theta(T, \lambda_1, \mu)\} = 1.$$

(II) Let $X(\mathscr{G}^d(t))$ denote the length of the longest directed path in $\mathscr{G}^d(t)$. If

$$\lambda_2 \bar{\theta}(\lambda_1, \mu) > 4 \log 3, \tag{1.4}$$

then with

$$\gamma = \max_{T>0} \left\{ \left(1 - \frac{4 \log 3}{\lambda_2 \theta(T, \lambda_1, \mu)} \right) e^{-\lambda_1 T} \right\} \tag{1.5}$$

we have

$$\mathbf{P}\{X(\mathscr{G}^d(t)) \geqslant \gamma\,|V(t)|\} \to 1 \qquad \text{as} \quad t \to \infty. \tag{1.6}$$

Remark 1.1. It will be seen from the proof, that under the condition (1.4) a long path can be composed entirely of those edges whose age is not greater than a constant T_0, such that

$$\gamma = \left(1 - \frac{4 \log 3}{\lambda_2 \theta(T_0, \lambda_1, \mu)}\right) e^{-\lambda_1 T_0}. \tag{1.7}$$

This implies that the process of link removal, regarded usually in a negative sense as a "permanent random damage" to a network (e.g., Dorogovtsev and Mendes[16]), may have, in fact, a positive effect on the efficiency of the network. Namely, there is no need to preserve all the connections in order to maintain a giant component as long as the system shows sufficient growth. Also, it is clear that unlike the scale-free networks (Albert *et al.*[17]) our model is robust to the deletion of any $o(e^{\lambda_1 t})$ number of nodes, since the average degree of any vertex is bounded by a constant $2\lambda_2/\mu$.

Next we find the limit for the expected value of the number of the directed cycles in the graph $\mathscr{G}^d(t)$. For any $k \geqslant 3$ and for any k different vertices $v^1,\ldots, v^k$ we shall call a set

$$C(v^1,\ldots, v^k) = \{(v^1, v^2), (v^2, v^3),\ldots, (v^k, v^1)\} \tag{1.8}$$

a k-cycle on these vertices.

Theorem 1.2. Let $C_k(t)$ denote the number of directed k-cycles in the graph $\mathscr{G}^d(t)$. If $\mu > 0$ then for any fixed $k \geqslant 3$

$$\begin{aligned}\mathbf{C}_k(\lambda_2, \mu, \lambda_1) &:= \lim_{t \to \infty} \mathbf{E} C_k(t)\\ &= \frac{1}{k} \lambda_2^k \mathbf{E} \prod_{i=1}^{k} \theta\left(\frac{1}{\lambda_1}(\eta_i \wedge \eta_{i+1}), \lambda_1, \mu\right) \exp\{\eta_i \wedge \eta_{i+1}\},\end{aligned} \tag{1.9}$$

where $\eta_1,\ldots, \eta_k$ are independent random variables with a common Exp(1) distribution, and $\eta_{k+1} \equiv \eta_1$.

This is the first exact result on the asymptotic structure of a non-homogeneous random graph model.

Remark 1.2. It will be seen in the proof below that the probability of an edge in our graph is bounded from above by $const/|V(t)|$. Thus using the same argument as in random graph theory (see Ballobás,[9] p. 78) we get the following statement. Although our formula (1.9) concerns the total number of cycles, in fact, these cycles almost surely have disjoint sets of vertices. Clearly, this refers only to the cycles of a fixed length.

Notice that information on cycles is of a particular interest for the applications, since the cycles are the most stable structures under certain dynamics in the neural networks (see, e.g., study by Xing and Gerstein,[14] and Turova[18]).

Remark 1.3. Formula (1.9) has a particularly simple form in the case $\lambda_1 = \mu$, namely

$$\lim_{t \to \infty} \mathbf{E}C_k(t) = \frac{1}{k}\left(\frac{\lambda_2}{\mu}\right)^k \mathbf{E} \prod_{i=1}^{k} \eta_i \wedge \eta_{i+1}. \tag{1.10}$$

Remark 1.4. For the non-directed paths and cycles statements (1.4), (1.5) and (1.9), (1.10), remain valid with a replacement of λ_2 by $2\lambda_2$.

Observe the monotonicity property of our model with respect to the parameter λ_2. Namely, if we fix the parameters λ_1 and μ, the connectivity of the graphs increases with λ_2. Thus one expects that there is a critical value, call it $\lambda_2^{\mathrm{cr}} = \lambda_2^{\mathrm{cr}}(\lambda_1, \mu)$, over which asymptotically almost every graph has a giant component whose order is proportional to the order of the graph, and below which almost every graph $\mathcal{G}(t)$ does not have such component. The first statement of Theorem 1.1 gives us an upper bound for λ_2^{cr}:

$$\lambda_2^{\mathrm{cr}} < \frac{\mu}{2}\left(\frac{\lambda_1}{\mu}\right)^{\frac{\lambda_1/\mu}{\lambda_1/\mu - 1}}. \tag{1.11}$$

Furthermore, Theorem 1.2 makes plausible the following conjecture on the exact value of λ_2^{cr}. Recall the well-known result on the model $G_{n,p}$ with $p = c/n$. It is proved (see Janson *et al.*[10]) that in the supercritical case $c > 1$ a substantial proportion of the vertices of the giant component of $G_{n,p}$ belongs to cycles. Hence, conjecturing that the giant component in our case has a similar structure, one may expect that the critical value λ_2^{cr} equals to

$$\tilde{\lambda}_2^{\mathrm{cr}} := \sup\left\{ x > 0 : \sum_{k=3}^{\infty} \mathbf{C}_k(2x, \mu, \lambda_1) < \infty \right\},$$

which by the formula of Theorem 1.2 can be found as the smallest positive root of the following hypergeometric function

$$1+\sum_{n=1}^{\infty} \frac{1}{n!}\left(-\frac{2x}{\mu}\right)^n \prod_{l=1}^{n}\left(\frac{1}{1+(l-1)\,\mu/\gamma}\right)=0\,.$$

Notice that as an easy corollary of Theorem 1.2 one gets

$$\tilde{\lambda}_2^{\mathrm{cr}} > \frac{\mu}{2}\sqrt{1+\frac{\lambda_1}{\mu}}.$$

The first statement of Theorem 1.1 improves Theorem 3 of Malyshev.[2] His result states under the conditions $2\lambda_2 > \mu$ and $\lambda_1 \ll \lambda_2$, the existence of a giant connected component in the limiting non-directed graph associated with the graph $(V(t), \mathscr{L}^1(t) \cup \mathscr{L}(t))$, which includes the type 1 edges. Moreover, due to Theorem 2 of Malyshev,[2] the condition $\lambda_1+2\lambda_2>\mu$ is necessary for the existence *a.s.* of a giant component in the graph including type 1 edges. The proofs of Malyshev[2] use mainly the mean values of graph characteristics to indicate the area of parameters where the random graph theory is valid.

Here we construct an approximate model in order to obtain the asymptotics and the rates of convergence for the probabilities of the edges. Notice that to get only the asymptotics of the conditional probability of any edge is a relatively easy task. This can be done using queuing theory. However, the major difficulty is to pass to the unconditional probabilities taking into account the growing (in time) number of the "small contributions." Our approach via discrete approximation allows us to control this situation by an appropriate scaling. Also, it enables us to find a homogeneous subgraph, and therefore to place our analysis within the framework of the theory of random graphs in the proof of Theorem 1.1. To prove Theorem 1.2 we exploit the non-homogeneity of the model. Note that our method is not limited to the Yule process $|V(t)|$, and can be applied to different random structures as long as the dynamics of the edges is governed by the independent Poisson processes. The later seem to be a necessary constraint in order to have (conditional) independence of edges.

A challenging problem for future study is to describe the self-organizing behaviour of the dynamical graphs, where the degree of a vertex depends on a local history of this vertex itself. A related static model of percolation on a triangle lattice was treated analytically by Jonasson[19] and Häggström and Turova.[20] However, for a dynamical model only computational results for a finite graph are available at present (e.g., Jin *et al.*[13]).

Note that in the paper of Malyshev[2] other models with different dynamics were introduced and studied as well. Krikun[21] found the asymptotics for the mean values of the longest tree associated with the type 1 edges. We also remark here that, following ideas of Peres,[22] our graph model can be viewed as a certain tree-indexed Markov chain. This is especially the case when type 1 edges are included. This observation may aid the detailed study of phase transitions and other properties of the model. We shall develop this approach elsewhere.

The rest of the paper is organized as follows. After we define and study an approximation model in Section 2, we prove Theorem 1.1 and Theorem 1.2 in Sections 3 and 4, respectively.

2. APPROXIMATION

2.1. The Model

Let $0 < \Delta < 1$ be fixed arbitrarily. We define on the same probability space a graph process $G^{d,m}(t)$ whose set of vertices is $V(t)$ introduced above, but whose edges can be appended or deleted only at the moments $\Delta, 2\Delta, \ldots$, according to the following rule. Set $G^{d,m}(0) = \mathcal{G}^{d,m}(0)$ as in the original model. Let further $L(t)$ denote the set of the edges of the graph $G^{d,m}(t)$, and $L_v(t)$ denote for any $v \in V(t)$ the set of the outcoming edges from the vertex v at time t, so that

$$L(t) = \bigcup_{v \in V(t)} L_v(t).$$

Here we assume that pair (v, v') represents a directed edge from the vertex v to the vertex v'. We shall call v the beginning and v' the end of the edge (v, v').

Given the graph $G^{d,m}(n\Delta) = (V(n\Delta), L(n\Delta))$ for any fixed $n \geqslant 0$, define

$$G^{d,m}(t) = (V(t), L(n\Delta)), \qquad n\Delta < t < (n+1)\Delta.$$

Next we set for any $v \in V(n\Delta)$

$$L_v((n+1)\Delta) = L_v^{\text{old}}((n+1)\Delta) \cup L_v^{\text{new}}((n+1)\Delta), \tag{2.1}$$

where $L_v^{\text{old}}((n+1)\Delta)$ is a subset of $L_v(n\Delta)$ such that any element of $L_v(n\Delta)$ belongs also to $L_v^{\text{old}}((n+1)\Delta)$ with a probability $e^{-\mu\Delta}$, while the set $L_v^{\text{new}}((n+1)\Delta)$ consists of the new edges which are defined as follows. Recall that in the definition of $\mathcal{G}^{d,m}(t)$ we associate with any vertex

$v \in V(n\Delta)$ a Poisson process with intensity λ_2, call it Y_v, whose occurrences correspond to the moments of appearance of new edges (of type 2) from the vertex v. So we set here

$$|L_v^{\text{new}}((n+1)\,\Delta)| = \begin{cases} \sum_{i=1}^{Y_v((n+1)\,\Delta)-Y_v(n\Delta)} \xi_i, & \text{if } \; Y_v((n+1)\,\Delta)-Y_v(n\Delta)>0, \\ 0, & \text{otherwise}, \end{cases} \tag{2.2}$$

where ξ_i are *i.i.d.* Bernoulli random variables with the parameter $e^{-\mu\Delta}$. The end of any edge from $L_v^{\text{new}}((n+1)\,\Delta)$ is distributed uniformly over the set $V(n\Delta)\setminus\{v\}$, i.e., for any $v' \in V(n\Delta)\setminus\{v\}$ and $k \geqslant 2$

$$\mathbf{P}\{(v,v') \in L_v^{\text{new}}((n+1)\,\Delta) \mid |V(n\Delta)|=k\} = \frac{1}{k-1}. \tag{2.3}$$

Finally, define

$$G^{d,m}((n+1)\,\Delta) = (V((n+1)\,\Delta),\, L((n+1)\,\Delta)),$$

where

$$L((n+1)\,\Delta) = \bigcup_{v \in V(n\Delta)} L_v((n+1)\,\Delta)$$

with L_v provided by (2.1).

The rate of this approximation is given by the following proposition.

Proposition 2.1. There is a positive constant C such that for any $T>\Delta$, integers $0<\tau\leqslant s\leqslant t=[\frac{T}{\Delta}]$, and an event

$$\mathscr{B}_{s,\tau} = \{u_s \in V(s\Delta)\setminus V((s-1)\,\Delta),\, v_\tau \in V(\tau\Delta)\setminus V((\tau-1)\,\Delta)\}$$

one has

$$|\mathbf{P}\{(u_s,v_\tau) \in L(T) \mid |V((s-1)\,\Delta)| = \bar{V}_{s-1}, \mathscr{B}_{s,\tau}\}$$
$$-\mathbf{P}\{(u_s,v_\tau) \in \mathscr{L}(T) \mid |V((s-1)\,\Delta)| = \bar{V}_{s-1}, \mathscr{B}_{s,\tau}\}| \leqslant C\,\frac{\Delta}{\bar{V}_{s-1}}. \tag{2.4}$$

The proof of this result is straightforward but lengthy. For the sake of brevity we refer for its proof to Turova.[23]

2.2. Probabilities of Edges

Consider the graph $G^{d,m}(T)$. We shall label its edges as follows. Let $L(S,T)$ for any $0 \leqslant S \leqslant T$ denote the set of the edges which appeared at time S. We call them the edges of the Sth generation. Clearly,

$$L(T) = \bigcup_{n=1}^{\left[\frac{T}{\Delta}\right]} L(n\Delta, T), \tag{2.5}$$

and

$$L(n\Delta, T) = \bigcup_{v \in V((n-1)\Delta)} L_v(n\Delta, T)$$

for any $n > 0$, where $L_v(S,T)$ denotes the subset of the edges of $L(S,T)$ outcoming from v.

Remark 2.1. Conditionally on $v \in V(n\Delta)$ the random sets $L_v(s\Delta, t\Delta)$ and $L_v(s'\Delta, t\Delta)$ are independent for any $n < s < s' \leqslant t$. If in addition, we condition on $v' \in V(n'\Delta)$, where $n' \geqslant n$ and $v' \neq v$, then the sets $L_v(s\Delta, t\Delta)$ and $L_{v'}(s\Delta, t\Delta)$ are independent for all $n' < s \leqslant t$, and $|L_v(s\Delta, t\Delta)| =_d |L_{v'}(s\Delta, t\Delta)|$.

Proposition 2.2. Conditionally on $v \in V((n-1)\Delta)$ where $0 < n \leqslant t$, the number of the edges $|L_v(n\Delta, t\Delta)|$ follows $\mathrm{Po}(\Delta\lambda_2 e^{-\mu\Delta(1+t-n)})$- distribution.

Proof. The proof is an immediate observation that according to the definition (2.2) the number of edges appeared from the vertex v at time $n\Delta$, follows $\mathrm{Po}(\lambda_2 \Delta e^{-\mu\Delta})$-distribution. This number we can consider as an increment of the Poisson process with intensity $\lambda_2 e^{-\mu\Delta}$ within a time-interval Δ. Then the process of deleting edges is equivalent to thinning of this Poisson process with probability $e^{-\mu\Delta}$. Since this thinning happens independently $t-n$ times (after each time interval Δ), the result follows. ∎

Next we shall derive formulae for the probabilities of the edges in $G^{d,m}(t)$. It is convenient for the simplicity of further notations to define also $V(t) = \varnothing = L(t)$ for any $t < 0$.

Proposition 2.3. Let $0 < \tau \leqslant s \leqslant t-1$ be integers, and assume that event

$$\begin{aligned}\mathscr{B}(\bar{V}) = \{&u_s \in V(s\Delta) \setminus V((s-1)\Delta), v_\tau \in V(\tau\Delta) \setminus V((\tau-1)\Delta),\\ &u_s \neq v_\tau, |V(k\Delta)| = \bar{V}_k, k = s, \ldots, t-1\}\end{aligned}$$

has a positive probability. Then

$$\mathbf{P}\{(u_s, v_\tau) \in L(t\Delta) \mid \mathscr{B}(\bar{V})\} = 1 - \prod_{k=s+1}^{t} \exp\left\{ -\Delta\lambda_2 e^{-\mu\Delta(1+t-k)} \frac{1}{\bar{V}_{k-1} - 1} \right\}. \tag{2.6}$$

Proof. Let us use a shorthand notation $\mathbf{P}_{\bar{V}}\{\cdot\} = \mathbf{P}\{\cdot \mid \mathscr{B}(\bar{V})\}$ for the conditional probability. Taking into account Remark 2.1 and observing that an edge between u_s and v_τ could not appear earlier than at time $((s \vee \tau) + 1)\,\Delta = (s+1)\,\Delta$, we derive according to (2.5):

$$\begin{aligned} \mathbf{P}_{\bar{V}}\{(u_s, v_\tau) \notin L(t\Delta)\} &= \mathbf{P}_{\bar{V}}\left\{ (u_s, v_\tau) \notin \bigcup_{k=s+1}^{t} L(k\Delta, t\Delta) \right\} \\ &= \prod_{k=s+1}^{t} \mathbf{P}_{\bar{V}}\{(u_s, v_\tau) \notin L_{u_s}(k\Delta, t\Delta)\}. \end{aligned} \tag{2.7}$$

Consider now for $s < k \leqslant t$

$$\begin{aligned} \mathbf{P}_{\bar{V}}\{(u_s, v_\tau) \notin L_{u_s}(k\Delta, t\Delta)\} = \sum_{n=0}^{\infty} &\mathbf{P}_{\bar{V}}\{(u_s, v_\tau) \notin L_{u_s}(k\Delta, t\Delta) \mid |L_{u_s}(k\Delta, t\Delta)| = n\} \\ &\times \mathbf{P}_{\bar{V}}\{|L_{u_s}(k\Delta, t\Delta)| = n\}. \end{aligned} \tag{2.8}$$

Notice that in order to fulfill the condition of the proposition we must have $\bar{V}_{k-1} \geqslant 2$, since at least $\{u_s, v_\tau\} \in V(s\Delta) \subseteq V((k-1)\,\Delta)$. Then according to the definition of the model (see (2.3)) we have

$$\mathbf{P}_{\bar{V}}\{(u_s, v_\tau) \notin L_{u_s}(k\Delta, t\Delta) \mid |L_{u_s}(k\Delta, t\Delta)| = n\} = \left(1 - \frac{1}{\bar{V}_{k-1} - 1}\right)^n. \tag{2.9}$$

Let $\nu(k, t, \Delta)$, $s < k \leqslant t$, be independent $\mathrm{Po}(\Delta\lambda_2 e^{-\mu\Delta(1+t-k)})$-distributed random variables whose probability generating functions we denote further by $g_{\nu(k,t,\Delta)}$. Recall, that by its definition

$$g_{\nu(k,t,\Delta)}(1-p) = \exp\{-\Delta\lambda_2 e^{-\mu\Delta(1+t-k)} p\}. \tag{2.10}$$

Due to Proposition 2.2 we have conditionally on $\mathscr{B}(\bar{V})$

$$|L_{u_s}(k\Delta, t\Delta)| =_d \nu(k, t, \Delta). \tag{2.11}$$

Substituting (2.9) into (2.8), and taking into account (2.11) we readily derive

$$\mathbf{P}_{\bar{V}}\{(u_s, v_\tau) \notin L_{u_s}(k\Delta, t\Delta)\} = g_{\nu(k,t,\Delta)}\left(1 - \frac{1}{\bar{V}_{k-1} - 1}\right), \tag{2.12}$$

which by (2.10) becomes

$$\mathbf{P}_{\bar{V}}\{(u_s, v_\tau) \notin L_{u_s}(k\Delta, t\Delta)\} = \exp\left\{-\Delta\lambda_2 \frac{e^{-\mu\Delta(1+t-k)}}{\bar{V}_{k-1}-1}\right\}. \tag{2.13}$$

Combination of (2.12) and (2.7) yields

$$\mathbf{P}_{\bar{V}}\{(u_s, v_\tau) \in L(t\Delta)\} = 1 - \prod_{k=s+1}^{t} g_{v(k,t,\Delta)}\left(1 - \frac{1}{\bar{V}_{k-1}-1}\right), \tag{2.14}$$

which together with the formula (2.10) immediately implies the statement of the proposition. ▮

It is easy to see that the properties of the Poisson process imply the conditional independence of the edges in our model. More exactly, we state it as follows.

Proposition 2.4. Assume that for some $k > 0$, $t \in \mathbf{Z}_+$, and $0 \leqslant s_i < t$, $i = 0, 1, \ldots, k$, the event

$$\begin{aligned} \mathscr{A}_{\bar{V}} := \{ & v^i_{s_i} \in V(s_i\Delta) \setminus V((s_i - 1)\,\Delta),\ 0 \leqslant i \leqslant k, \\ & |V(n\Delta)| = \bar{V}_n,\ \min_{0 \leqslant i \leqslant k} s_i \leqslant n < t\}, \end{aligned}$$

where $v^i_{s_i}$, $i = 0, 1, \ldots, k$, are $k+1$ different vertices, has a positive probability. Then

$$\mathbf{P}\{(v^0_{s_0}, v^i_{s_i}) \in L(t\Delta), i = 1, \ldots, k \mid \mathscr{A}_{\bar{V}}\} = \prod_{i=1}^{k} \mathbf{P}\,\{(v^0_{s_0}, v^i_{s_i}) \in L(t\Delta) \mid \mathscr{A}_{\bar{V}}\}. \tag{2.15}$$

Propositions 2.3 and 2.4 obviously yield the following result. Let us call the vertices of the set $V(s) \setminus V(s - \Delta)$ the vertices of the sth generation.

Corollary 2.1. Conditionally on an event $\{|V(k\Delta)| = \bar{V}_k,\ 0 \leqslant k \leqslant t-1\}$ the edges between the vertices of the graph $G^d(t\Delta)$ are independent, and for any two vertices u_s and v_τ of the Δsth and $\Delta\tau$th generations, respectively, the probability that there is an edge from u_s to v_τ is given by

$$\begin{aligned} & \mathbf{P}\{(u_s, v_\tau) \in L(\Delta t) \mid \{u_s, v_\tau\} \subseteq V((t-1)\,\Delta), |V(k\Delta)| = \bar{V}_k, 0 \leqslant k \leqslant t-1\} \\ & \quad = 1 - \prod_{k=(\tau \vee s)+1}^{t} \exp\left\{-\Delta\lambda_2 e^{-\mu\Delta(1+t-k)} \frac{1}{\bar{V}_{k-1}-1}\right\}. \end{aligned} \tag{2.16}$$

The conditional independence of the edges in the original model follows by the general properties of the Poisson process as well.

Proposition 2.5. Conditionally on the trajectory $\{|V(S)| = \bar{V}(S), 0 \leqslant S \leqslant T\}$, the edges of the graph $\mathscr{G}^d(T)$ are independent.

2.3. Scaling

Let $S \in \mathbf{R}_+$ and $T > \Delta$ be fixed arbitrarily, and set $t = [\frac{T}{\Delta}]$. Here we shall find the "most probable trajectory" of $\{|V(S+k\Delta)|, k = 0,..., t\}$. Define an event

$$\mathscr{A}(S, T, \Delta) = \left\{ \left| \frac{|V(S+(k+1)\,\Delta)|}{|V(S+k\Delta)|} - (1+\lambda_1 \Delta) \right| \leqslant \Delta^{3/2}, 0 \leqslant k \leqslant \left[\frac{T}{\Delta}\right] - 1 \right\}. \tag{2.17}$$

The following lemma will help us to choose later on a proper scaling of Δ (and T) with respect to S, so that

$$\mathbf{P}\{\mathscr{A}(S, T, \Delta)\} \to 1$$

as $S \to \infty$ (and $T \to \infty$) while $\Delta \to 0$ simultaneously.

Lemma 2.1. For any $\bar{V}_0 \geqslant 1$

$$\mathbf{P}\{\mathscr{A}(S, T, \Delta) \mid |V(S)| = \bar{V}_0\} \geqslant \left(1 - \frac{\lambda_1}{\Delta^2 \bar{V}_0}\right)^{\left[\frac{T}{\Delta}\right]}. \tag{2.18}$$

Proof. Notice, that given $|V(S+k\Delta)| = \bar{V}_k$ with $\bar{V}_k > 0$

$$|V(S+(k+1)\,\Delta)| =_d \sum_{i=1}^{\bar{V}_k} (1+\eta_i),$$

where η_i, $i \geqslant 1$, are $\mathrm{Po}(\lambda_1 \Delta)$-distributed independent random variables. Hence, due to the Chebyshev's inequality we have for any $k \geqslant 0$, $\bar{V}_k \geqslant \bar{V}_0$ and $\delta = \Delta^{3/2}$

$$\mathbf{P}\left\{ \left| \frac{|V(S+(k+1)\,\Delta)|}{|V(S+k\Delta)|} - (1+\lambda_1 \Delta) \right| > \delta \;\middle|\; |V(S+k\Delta)| = \bar{V}_k \right\} \leqslant \frac{\lambda_1 \Delta}{\delta^2 \bar{V}_k} \leqslant \frac{\lambda_1 \Delta}{\delta^2 \bar{V}_0}. \tag{2.19}$$

Since $|V(S+k\Delta)|$, $k \geqslant 0$, is a Markov chain, this allows us to derive

$$\mathbf{P}\left\{\left|\frac{|V(S+(k+1)\,\Delta)|}{|V(S+k\Delta)|}-(1+\lambda_1\Delta)\right| \leqslant \delta, 0 \leqslant k \leqslant t-1 \,\middle|\, |V(S)|=\bar{V}_0\right\}$$

$$\geqslant \left(1-\frac{\lambda_1\Delta}{\delta^2\bar{V}_0}\right)$$

$$\times \mathbf{P}\left\{\left|\frac{|V(S+(k+1)\,\Delta)|}{|V(S+k\Delta)|}-(1+\lambda_1\Delta)\right| \leqslant \delta, 0 \leqslant k \leqslant t-2 \,\middle|\, |V(S)|=\bar{V}_0\right\}$$

$$\geqslant \cdots \geqslant \left(1-\frac{\lambda_1\Delta}{\delta^2\bar{V}_0}\right)^t. \tag{2.20}$$

The statement (2.18) follows. ∎

3. PROOF OF THEOREM 1.1

We begin with the proof of the second statement (II) of Theorem 1.1. The idea is to find an "optimal," i.e., with a high probability of edges, homogeneous subgraph and use the known results from the random graph theory. Let $T>0$ be fixed arbitrarily. Consider for any $S>1$ a subgraph $G_S^{d,m}(S+T)$ obtained from $G^{d,m}(S+T)$ by removing the edges which appeared before the time S, and removing the vertices (together with the adjacent edges) which appeared after the time S. Thus $V(S)$ is the set of vertices of $G_S^{d,m}(S+T)$. Clearly, for all vertices of this subgraph the distributions of outcoming edges are identical.

The bound (2.18) allows us to derive for any event $\mathscr{B}$

$$\mathbf{P}\{\mathscr{B}\} \geqslant \sum_{\bar{V}_0 \geqslant S} \mathbf{P}\{\mathscr{B} \mid \mathscr{A}(S,T,\Delta), |V(S)|=\bar{V}_0\}\left(1-\frac{\lambda_1}{\Delta^2\bar{V}_0}\right)^{\left[\frac{T}{\Delta}\right]} \mathbf{P}\{|V(S)|=\bar{V}_0\}. \tag{3.1}$$

Let us fix now an event

$$\bar{V} := \{|V(S+s)|=\bar{V}_s, 0 \leqslant s \leqslant T\} \in \mathscr{A}(S,T,\Delta), \tag{3.2}$$

and conditionally on this event consider graph $G_S^{d,m}(S+t\Delta)$. As we have noticed, the structure of this graph is homogeneous. Therefore given $|V(S)|=\bar{V}_0$ we can define $V(S)$ to be simply a set of $\bar{V}_0$ different elements. Let G_S^d be a directed graph associated with the multi-graph $G_S^{d,m}$. In the sequel we shall use the following short-hand notation for the small terms (e.g., for the functions which converge to zero as $\Delta \to 0$). Denote $\varepsilon(x)$ any function which decays at zero so that

$$|\varepsilon(x)| \leqslant C_\varepsilon x, \qquad 0 \leqslant x < 1, \tag{3.3}$$

where positive constant C_ε is independent of $S, T, \bar{V}$ and Δ.

Lemma 3.1. Conditionally on $\bar{V}$ and an event $\{u, v \in V(S)\}$, the probability that there is an edge from u to v in the graph $G_S^d(S+T)$ is

$$p_{d,\bar{V}}^G(T) = 1 - \exp\left\{ -\frac{\lambda_2(\theta(T, \lambda_1, \mu) + \varepsilon(\Delta))}{\bar{V}_0} (1 + \varepsilon(\Delta^{1/2}))(1 + \varepsilon(\bar{V}_0^{-1})) \right\}, \tag{3.4}$$

with function θ defined in (1.1).

Proof. Let $t = [\frac{T}{\Delta}]$ assuming without loss of generality that $0 < \Delta < \min\{1, T\}/2$. Using formula (2.13) and conditional independence of the edges we derive

$$\begin{aligned} p_{d,\bar{V}}^G(T) &\equiv \mathbf{P}\left\{ (u, v) \in \bigcup_{k=2}^{t} L_u\left(\left[\frac{S}{\Delta}\right]\Delta + k\Delta, S+T \right) \middle| u, v \in V(S), \bar{V} \right\} \\ &= 1 - \prod_{k=2}^{t} \exp\left\{ -\Delta \lambda_2 e^{-\mu \Delta(1+t-k)} \frac{1}{\bar{V}_{(S-\Delta[\frac{S}{\Delta}]) + \Delta(k-1)} - 1} \right\}. \end{aligned} \tag{3.5}$$

Observe that given (3.2) we have by the definition (2.17)

$$\frac{1}{\bar{V}_{(S-\Delta[\frac{S}{\Delta}]) + \Delta k} - 1} = \frac{e^{-\lambda_1 \Delta k}}{\bar{V}_0} (1 + \varepsilon(k\Delta^{3/2}))(1 + \varepsilon(\bar{V}_0^{-1})) \tag{3.6}$$

for any $0 < k < t = [\frac{T}{\Delta}]$. Substituting (3.6) into (3.5), we obtain (3.4) after simple calculation. ∎

Let also $\mathscr{G}_S^{d,m}(S+T)$ be a subgraph obtained from $\mathscr{G}^{d,m}(S+T)$ by removing the edges which appeared before the time S, and removing the vertices (together with the adjacent edges) which appeared after the time S. Again, $V(S)$ is the set of the vertices of graph $\mathscr{G}_S^{d,m}(S+T)$. Clearly, the graph $\mathscr{G}_S^d$ associated with $\mathscr{G}_S^{d,m}$ is also homogeneous. Hence we can denote for any $u, v \in V(S)$ the probability of a directed edge from u to v in the graph $\mathscr{G}_S^d(S+T)$ as

$$p_{d,\bar{V}}(T) = \mathbf{P}\left\{ (u, v) \in \bigcup_{k=1}^{t+1} \mathscr{L}_v(S + k\Delta, S+T) \middle| u, v \in V(S), \bar{V} \right\}.$$

It is easy to see following the proof of (2.4) that

$$|p_{d,\bar{V}}(T)-p^G_{d,\bar{V}}(T)| \leqslant \frac{C\Delta}{\bar{V}_0} \tag{3.7}$$

for some constant C. Combining this and (3.4) we obtain for any fixed constant $c>1$ and all sufficiently small Δ and large $\bar{V}_0$

$$p_{d,\bar{V}}(T) \geqslant p^G_{d,\bar{V}}(T)-\frac{C\Delta}{\bar{V}_0} \geqslant \frac{\lambda_2\theta(T,\lambda_1,\mu)}{c\bar{V}_0} =: p. \tag{3.8}$$

On the same probability space let $\vec{G}_{n,p}$ be a directed random graph with n vertices, and with a probability p of the edges. For any directed graph G let $X(G)$ denote the length of the longest directed path in G. Recall a known result (see Theorem 2, p. 185, Bollobás[9]) that if $0<pn<\log n-3\log\log n$ then asymptotically almost every graph $\vec{G}_{n,p}$ contains a directed path of length at least $(1-\frac{4\log 2}{pn})\,n$, i.e.,

$$\mathbf{P}\left\{X(\vec{G}_{n,p}) \geqslant \left(1-\frac{4\log 2}{pn}\right)n\right\} \to 1 \qquad \text{as} \quad n\to\infty. \tag{3.9}$$

Obviously, due to (3.8) and the conditional independence of the edges of $\mathscr{G}^d$, we have for any $N>0$ and all sufficiently small Δ and large $\bar{V}_0$

$$\mathbf{P}_{\bar{V}}\{X(\mathscr{G}^d_S(S+T)) \geqslant N\} \geqslant \mathbf{P}\{X(\vec{G}_{\bar{V}_0,p}) \geqslant N\}. \tag{3.10}$$

Now we derive from (3.9) and (3.10) with the value of p from (3.8)

$$\mathbf{P}_{\bar{V}}\left\{X(\mathscr{G}^d_S(S+T)) \geqslant \left(1-\frac{4c\log 2}{\lambda_2\theta(T,\lambda_1,\mu)}\right)\bar{V}_0\right\} \to 1 \qquad \text{as} \quad \bar{V}_0\to\infty. \tag{3.11}$$

for all sufficiently small Δ. Notice that given (3.2) we have by the definition (2.17)

$$\bar{V}_T=\bar{V}_0 e^{\Delta\lambda_1 t}(1+\varepsilon(\Delta^{1/2})).$$

Substituting this into (3.11) we obtain

$$\mathbf{P}_{\bar{V}}\left\{X(\mathscr{G}^d_S(S+T)) \geqslant \left(1-\frac{4\log 3}{\lambda_2\theta(T,\lambda_1,\mu)}\right)e^{-\lambda_1 T}\bar{V}_T\right\} > 1-\epsilon_0(\bar{V}_0)$$

for all sufficiently small $\varDelta$, where $\epsilon_0(\bar{V}_0)$ is some positive function such that $\epsilon_0(\bar{V}_0) \to 0$ as $\bar{V}_0 \to \infty$. The last bound together with (3.2) implies

$$\mathbf{P}\left\{X(\mathscr{G}_S^d(S+T)) \geqslant \left(1-\frac{4\log 3}{\lambda_2 \theta(T, \lambda_1, \mu)}\right) e^{-\lambda_1 T}\,|V(S+T)| \,\middle|\, \mathscr{A}(S, T, \varDelta), |V(S)| = \bar{V}_0\right\} > 1-\epsilon_0(\bar{V}_0)$$

for all large $\bar{V}_0$ and small $\varDelta$. Combination of the last bound and (3.1) allows us to derive:

$$\begin{aligned}\mathbf{P}&\left\{X(\mathscr{G}_S^d(S+T)) \geqslant \left(1-\frac{4\log 3}{\lambda_2 \theta(T, \lambda_1, \mu)}\right) e^{-\lambda_1 T}\,|V(S+T)|\right\}\\ &\geqslant \sum_{\bar{V}_0 \geqslant S} (1-\epsilon_0(\bar{V}_0))\left(1-\frac{\lambda_1}{\varDelta^2 \bar{V}_0}\right)^{\frac{T}{\varDelta}} \mathbf{P}\{|V(S)| = \bar{V}_0\}\\ &\geqslant (1-\sup_{x \geqslant S}\, \epsilon_0(x))\left(1-\frac{\lambda_1}{\varDelta^2 S}\right)^{\frac{T}{\varDelta}} \mathbf{P}\{|V(S)| \geqslant S\}.\end{aligned} \tag{3.12}$$

Making use of the known results on Yule process (see, e.g., Athreya and Ney,[12] p. 109) one has a formula

$$\mathbf{P}\{|V(S)| \geqslant k\} = (1-e^{-\lambda_1 S})^{k-1}, \qquad k \geqslant 1, \quad S > 0. \tag{3.13}$$

Letting now $\varDelta = S^{-1/4}$, it is easy to see that the right-hand side of (3.12) goes to one as $S \to \infty$. Obviously, for any $N > 1$

$$\mathbf{P}\{X(\mathscr{G}^d(S+T)) \geqslant N\} \geqslant \mathbf{P}\{X(\mathscr{G}_S^d(S+T)) \geqslant N\} \tag{3.14}$$

since $\mathscr{G}_S^d(S+T)$ is a subgraph of $\mathscr{G}^d(S+T)$. Hence statement (II) of Theorem 1.1 follows from (3.12) as soon as we choose $T = T_0$ to satisfy (1.7).

Now we shall turn to the proof of the first statement of Theorem 1.1. Here we choose T so that $\theta(T, \lambda_1, \mu) = \bar{\theta}(\lambda_1, \mu)$ (recall condition (1.3)). Consider now $\mathscr{G}_S(S+T)$ for $S > 1$. Let $p_{\bar{V}}(T)$ denote the probability of edge between two arbitrary vertices u and v in the graph $\mathscr{G}_S(S+T)$. Clearly, due to the conditional independence of the edges we have

$$p_{\bar{V}}(T) = 1-(1-p_{d,\bar{V}}(T))^2,$$

as well as

$$p_{\bar{V}}^G(T) = 1-(1-p_{d,\bar{V}}^G(T))^2, \tag{3.15}$$

which allows us to derive with a help of (3.7)

$$|p_{\bar{V}}(T) - p_{\bar{V}}^{\mathcal{G}}(T)| \leqslant 3C\,\frac{\Delta}{\bar{V}_0}.$$

Then we derive from (3.4) and (3.15) that for any fixed $c' > 1$ and for all sufficiently small Δ and large $\bar{V}_0$

$$p_{\bar{V}}(T) \geqslant p_{\bar{V}}^{\mathcal{G}}(T) - 3C\,\frac{\Delta}{\bar{V}_0} \geqslant \frac{2\lambda_2\theta(T, \lambda_1, \mu)}{c'\bar{V}_0} = \frac{2\lambda_2\bar{\theta}(\lambda_1, \mu)}{c'\bar{V}_0},$$

which under the condition (1.3) proves the existence of some constant $p' > 1$ such that for all sufficiently small Δ and large $\bar{V}_0$

$$p_{\bar{V}}(T) \geqslant \frac{p'}{\bar{V}_0}. \tag{3.16}$$

Let $G_{n,p}$ denote a random graph with n vertices and with a probability $p = p'/n$ of the edges. Set further $M(G)$ to be the order of the largest component in a graph G. Due to Theorem 5.4 by Janson *et al.*,[10] p. 109, if $p' > 1$ then

$$\mathbf{P}\{M(G_{n,p}) \geqslant (1+o(1))\,\beta n\} \to 1 \qquad \text{as} \quad n \to \infty, \tag{3.17}$$

where positive constant β satisfies the equation

$$\beta + e^{-\beta p'} = 1.$$

Clearly, together with (3.16) this implies for all sufficiently small Δ

$$\begin{aligned} &\mathbf{P}_{\bar{V}}\{M(\mathcal{G}_S(S+T_1)) \geqslant (1+o(1))\,\beta\bar{V}_0\} \\ &\quad \geqslant \mathbf{P}\{M(G_{\bar{V}_0,p}) \geqslant (1+o(1))\,\beta\bar{V}_0\} \to 1 \end{aligned} \tag{3.18}$$

as $\bar{V}_0 \to \infty$. Then we derive the first statement (I) of Theorem 1.1 using (3.18), just by repeating the same argument as we derived the second statement using (3.11). ▮

Remark 1.1 is merely due to the formulae (3.12) and (3.14) in the above proof.

4. PROOF OF THEOREM 1.2

Consider graph $\mathcal{G}^d(S+T)$, where $S > 0$ and $T > 0$. Let $k \geqslant 3$ be fixed arbitrarily, and let $C_k^{(S)}(S+T)$ denote the number of k-cycles composed entirely of the edges whose age is less than T. Let also $C_k^{(<S)}(S+T)$ be the number of k-cycles which contain at least one edge of the age greater or equal T. Thus the total number of the k-cycles in the graph $\mathcal{G}^d(S+T)$ is

$$C_k(S+T) = C_k^{(S)}(S+T) + C_k^{(<S)}(S+T). \tag{4.1}$$

From now on assume that

$$S > 1, \qquad T = S^{3/2}, \qquad \Delta = S^{-3(k+2)/2}, \tag{4.2}$$

and for these values we define $\mathscr{A}(S) = \mathscr{A}(S, T, \Delta)$ as in (2.17). Recall that according to (2.18)

$$\mathbf{P}\{\mathscr{A}(S) \mid |V(S)| = \bar{V}_0\} \geqslant \left(1 - \frac{\lambda_1}{S^{-3(k+2)}\bar{V}_0}\right)^{S^{3(k+3)/2}}.$$

This yields

$$\begin{aligned} \mathbf{P}\{\mathscr{A}(S)\} &\geqslant \left(1 - \frac{\lambda_1}{S^{3/2}}\right)^{S^{3/2+3(k+2)/2}} \mathbf{P}\{|V(S)| > S^{3(k+2)+3/2}\} \\ &\geqslant \left(1 - \frac{c}{S^{3(k+2)/2}}\right) \mathbf{P}\{|V(S)| > S^{3(k+2)+3/2}\}, \end{aligned}$$

where c is some positive constant independent of S. This bound in combination with (3.13) proves that

$$\mathbf{P}\{\mathscr{A}(S)\} \to 1 \qquad \text{as} \quad S \to \infty. \tag{4.3}$$

Observe, that the following equality holds

$$\begin{aligned} \lim_{S\to\infty} \mathbf{E}C_k(S) &= \lim_{S\to\infty} \mathbf{E}C_k^{(S)}(S+S^{3/2})\,\mathbf{1}_{\mathscr{A}(S)} \\ &\quad + \lim_{S\to\infty} \mathbf{E}C_k^{(<S)}(S+S^{3/2})\,\mathbf{1}_{\mathscr{A}(S)} + \lim_{S\to\infty} \mathbf{E}C_k(S+S^{3/2})\,\mathbf{1}_{\Omega\setminus\mathscr{A}(S)}, \end{aligned} \tag{4.4}$$

provided the existence of all three limits in the right-hand side; $\mathbf{1}_B$ denotes here an indicator function of an event B. In the following we shall study the last three terms separately. In fact, we will show that only the first term in the right-hand side of (4.4) provides a non-zero contribution while the last two limits exist and equal zero.

Let us fix $\lambda_1 > 0$ and $\mu > 0$ arbitrarily, and for the simplicity of notations we shall write since now on $\theta(T) = \theta(T, \lambda_1, \mu)$.

Lemma 4.1. For any fixed finite k

$$\lim_{S\to\infty} \mathbf{E}C_k^{(S)}(S+S^{3/2})\,\mathbf{1}_{\mathscr{A}(S)}$$

$$=\frac{1}{k}\lambda_2^k\int_0^\infty\cdots\int_0^\infty\left(\prod_{i=1}^k e^{-s_i}\exp\{(s_i\wedge s_{i+1})\}\,\theta\left(\frac{s_i}{\lambda_1}\wedge\frac{s_{i+1}}{\lambda_1}\right)\right)ds_1\cdots ds_k. \tag{4.5}$$

Before we proceed with a proof of this lemma we shall obtain some useful results on the probabilities of edges. Conditionally on the trajectory

$$\bar{V}=\{|V(S+s)|=\bar{V}_s,\ 0\leqslant s\leqslant T\}\in\mathscr{A}(S), \tag{4.6}$$

we define $\delta_0(V)=V(S)$ to be simply a set of $|\delta_0(V)|=\bar{V}_0$ different elements. We shall also define conditionally on $\bar{V}$

$$\delta_i(V)=V(S+\Delta i)\setminus V(S+\Delta(i-1)),\qquad 1\leqslant i<\left[\frac{T}{\Delta}\right]=:t,$$

and $\delta_t(V)=V(S+T)\setminus V(S+\Delta(t-1))$ to be the sets of vertices labeled by the moments of appearance provided by the condition $\bar{V}$. The cardinalities of these sets are $|\delta_i(V)|=\bar{V}_{i\Delta}-\bar{V}_{(i-1)\Delta}$, and $|\delta_t(V)|=\bar{V}_T-\bar{V}_{(t-1)\Delta}$, respectively.

Let us denote here $\mathscr{L}((s,s'],t)\subseteq\mathscr{L}(t)$ the subset of the edges of graph $\mathscr{G}^{d,m}(t)$ which appeared during the time interval $(s,s'\wedge t]$. Consider now the following subgraph

$$\widetilde{\mathscr{G}}_S^{d,m}(S+T):=(V(S+T),\mathscr{L}((S,S+T],S+T)) \tag{4.7}$$

obtained from $\mathscr{G}^{d,m}(S+T)$ by removing the edges whose age is greater or equal T. Notice that this subgraph is defined on the entire set of vertices $V(S+T)$, which conditionally on $\bar{V}$ is $V(S+T)=\bigcup_{i=0}^t\delta_i(V)$.

Let $u_s\in\delta_s(V)$ and $v_\tau\in\delta_\tau(V)$, where $0\leqslant\tau,s\leqslant t$. Conditionally on $\bar{V}$ let us denote here $p^G(u_s,v_\tau)$ and $p(u_s,v_\tau)$ the probabilities of the edge from u_s to v_τ in the graphs $G^d(S+T)$ and $\mathscr{G}^d(S+T)$, correspondingly. Let also $\tilde{p}(u_s,v_\tau)$ denote the probability of the edge from u_s to v_τ in the graph $\widetilde{\mathscr{G}}_S^d(S+T)$. Substituting (3.6) into (2.16) we derive in the case $s\vee\tau>0$:

$$p^G(u_s,v_\tau)=\frac{\lambda_2(\theta(T-(s\vee\tau)\,\Delta)+\varepsilon(\Delta))}{\bar{V}_{(s\vee\tau)\,\Delta}}(1+\varepsilon(\Delta^{1/2}T))(1+\varepsilon(\bar{V}_0^{-1})). \tag{4.8}$$

Notice that if $s\vee\tau>0$ then any edge between u_s and v_τ in the original graph $\mathscr{G}^{d,m}(S+T)$ has the age less than T, and therefore this edge belongs to the edges of the subgraph $\widetilde{\mathscr{G}}_S^{d,m}(S+T)$ as well. Thus combining (4.8) with (2.4) we obtain in the case $s \vee \tau > 0$

$$\begin{aligned} p(u_s, v_\tau) &= \tilde{p}(u_s, v_\tau) \\ &= \frac{\lambda_2(\theta(T-(s\vee\tau)\,\Delta)+\varepsilon(\Delta))}{\bar{V}_{(s\vee\tau)\,\Delta}}\,(1+\varepsilon(\Delta^{1/2}T))(1+\varepsilon(\bar{V}_0^{-1})) \\ &=: f((s\vee\tau)\,\Delta), \end{aligned} \tag{4.9}$$

where f is just a short-hand notation of the preceding function. Next observe that the probability of an edge between two arbitrary vertices u_0, v_0 of the subset $\delta_0(V)$ of the vertices of the graph $\widetilde{\mathscr{G}}_S^d(S+T)$ is

$$\tilde{p}(u_0, v_0) = 1 - \mathbf{P}_{\bar{V}}\{(u_0, v_0) \notin \mathscr{L}((S, S+T], S+T)\}.$$

Taking into account the rate of approximation and making use of formula (2.13) we derive from here

$$\tilde{p}(u_0, v_0) = \frac{\lambda_2(\theta(T)+\varepsilon(\Delta))}{\bar{V}_0}\,(1+\varepsilon(\Delta^{1/2}T))(1+\varepsilon(\bar{V}_0^{-1})) =: f(0). \tag{4.10}$$

For the further reference consider here also the probability $p(u_0, v_0)$ where $u_0, v_0 \in \delta_0(V)$, in the graph $\mathscr{G}^d(S+T)$. Due to the conditional independence of the edges we have

$$\begin{aligned} p(u_0, v_0) &= 1 - \mathbf{P}_{\bar{V}}\{(u_0, v_0) \notin \mathscr{L}((S, S+T], S+T)\} \\ &\quad \times \mathbf{P}_{\bar{V}}\{(u_0, v_0) \notin \mathscr{L}((0, S), S+T)\} \\ &= f(0) + (1-f(0))\,\mathbf{P}_{\bar{V}}\{(u_0, v_0) \in \mathscr{L}((0, S), S+T)\}. \end{aligned} \tag{4.11}$$

Using (2.13) we can easily obtain a uniform bound for the probabilities of edges whose age is greater or equal T:

$$\mathbf{P}_{\bar{V}}\{(u_0, v_0) \in \mathscr{L}((0, S), S+T)\} \leqslant Ae^{-\mu T}, \tag{4.12}$$

where $A > 0$ is a constant independent of T. This together with (4.11) clearly implies

$$p(u_0, v_0) \leqslant f(0) + Ae^{-\mu T}. \tag{4.13}$$

Proof of Lemma 4.1. For simplicity we shall use here notations (4.2). For any graph G and cycle C let $\{C \in G\}$ denote the event that graph

G contains this cycle. Then given (4.6) with $\bar{V}_0 > k$ we have in the notations (1.8)

$$\mathbf{E}_{\bar{V}} C_k^{(S)}(S+T) = \frac{1}{k} \sum_{(v^1,\dots,v^k):\, v^j \in \bigcup_{i=0}^t \delta_i(V)} \mathbf{P}_{\bar{V}} \{C(v^1,\dots,v^k) \in \widetilde{\mathscr{G}}_S^d(S+T)\}, \tag{4.14}$$

where $\mathbf{E}_{\bar{V}}$ is a conditional expectation with respect to the event $\bar{V}$, and the sum runs over the ordered sets of k different vertices. Due to the conditional independence of the edges we have

$$\mathbf{P}_{\bar{V}} \{C(v^1,\dots,v^k) \in \widetilde{\mathscr{G}}_S^d(S+T)\} = \tilde{p}(v^k, v^1) \prod_{i=1}^{k-1} \tilde{p}(v^i, v^{i+1}). \tag{4.15}$$

Let $\mathscr{P}_k$ denote the set of all permutations $(\pi(1),\dots,\pi(k))$, and let also $\pi(k+1) \equiv \pi(1)$. Further for any $\pi \in \mathscr{P}_k$ and $v = (v^1,\dots,v^k)$ we define $\pi(v) = (v^{\pi(1)},\dots,v^{\pi(k)})$. Now we can rewrite (4.14) as

$$\begin{aligned}\mathbf{E}_{\bar{V}} C_k^{(S)}&(S+T)\\ &= \frac{1}{k} \sum_{0 \leqslant n_k \leqslant \cdots \leqslant n_1 \leqslant t} \; \sum_{\{v^1,\dots,v^k\}:\, v^i \in \delta_{n_i}(V)} \; \sum_{\pi \in \mathscr{P}_k} \mathbf{P}_{\bar{V}} \{C(\pi(v)) \in \widetilde{\mathscr{G}}_S^d(S+T)\},\end{aligned} \tag{4.16}$$

where the second sum runs over the non-ordered sets of k different vertices. Then according to (4.15) together with (4.9) and (4.10) we derive:

$$\begin{aligned}\mathbf{E}_{\bar{V}} C_k^{(S)}&(S+T)\\ &= \frac{1}{k} \sum_{0 \leqslant n_k \leqslant \cdots \leqslant n_1 \leqslant t} \; \sum_{\{v^1,\dots,v^k\}:\, v^i \in \delta_{n_i}(V)} \; \sum_{\pi \in \mathscr{P}_k} \prod_{i=1}^{k} f((n_{\pi(i)} \vee n_{\pi(i+1)})\, \Delta)\\ &= \frac{1}{k} \binom{|\delta_0(V)|}{k} \sum_{\pi \in \mathscr{P}_k} \prod_{i=1}^{k} f(0)\\ &\quad + \frac{1}{k} \sum_{l=0}^{k-1} \binom{|\delta_0(V)|}{l} \sum_{n_k = \cdots = n_{k-l+1} = 0 < n_{k-l} < \cdots < n_1 \leqslant t} \; \prod_{i=1}^{k-l} |\delta_{n_i}(V)|\\ &\quad \times \sum_{\pi \in \mathscr{P}_k} \prod_{i=1}^{k} f((n_{\pi(i)} \vee n_{\pi(i+1)})\, \Delta)\\ &\quad + \frac{1}{k} \sum_{l=0}^{k-2} \binom{|\delta_0(V)|}{l} {\sum}' \sum_{\{v^1,\dots,v^k\}:\, v^i \in \delta_{n_i}(V)} \; \sum_{\pi \in \mathscr{P}_k} \prod_{i=1}^{k} f((n_{\pi(i)} \vee n_{\pi(i+1)})\, \Delta),\end{aligned} \tag{4.17}$$

where for each $0 \leqslant l \leqslant k-2$ the sum $\sum'$ runs over the vectors $(n_k, \dots, n_1)$ such that $n_k = \cdots = n_{k-l+1} = 0$ unless $l = 0$, and there are ties among the numbers $n_{k-l}, \dots, n_1$, i.e.,

$$n_k = \cdots = n_{k-l+1} = 0 < n_{k-l} \leqslant \cdots \leqslant n_1 \leqslant t:$$
$$n_i = n_j \qquad \text{for some} \quad 1 \leqslant i < j \leqslant k-l.$$

Note that (4.6) and (2.17) yield

$$|\delta_{n_i}(V)| = \bar{V}_{n_i \Delta} - \bar{V}_{(n_i - 1)\Delta} = \Delta \lambda_1 \bar{V}_{n_i \Delta}(1 + \varepsilon(\Delta^{1/2})), \tag{4.18}$$

and also

$$\bar{V}_{n_i \Delta} = \bar{V}_0 e^{\Delta \lambda_1 n_i}(1 + \varepsilon(n_i \Delta^{3/2})). \tag{4.19}$$

Substituting (4.18) into (4.17) we get

$$\begin{aligned} \mathbf{E}_{\bar{V}} C_k^{(S)}(S+T) = {} & \frac{1}{k}\binom{\bar{V}_0}{k} k!\, f^k(0) + \frac{1}{k}\sum_{l=0}^{k-1} \sum_{\pi \in \mathcal{P}_k} \Bigg\{ (1+\varepsilon(\Delta^{1/2}))\, \lambda_1^{k-l} \\ & \times \sum_{n_k = \cdots = n_{k-l+1} = 0 < n_{k-l} < \cdots < n_1 \leqslant t} \Delta^{k-l} \binom{\bar{V}_0}{l} \left(\prod_{i=1}^{k-l} \bar{V}_{n_i \Delta} \right) \\ & \times \prod_{i=1}^{k} f((n_{\pi(i)} \vee n_{\pi(i+1)})\, \Delta) + D_k(\pi, l) \Bigg\}, \end{aligned} \tag{4.20}$$

where $D_k(\pi, k-1) = 0$, and for all $0 \leqslant l \leqslant k-2$

$$D_k(\pi, l) = \sum_{m=1}^{k-l-1} \sum_{(i_1, \dots, i_m):\, i_1 + \cdots + i_m = k-l,\, i_j \geqslant 1} D_k(\pi, l, m, \bar{\imath}), \tag{4.21}$$

with

$$D_k(\pi, l, m, \bar{\imath}) = \sum \binom{\bar{V}_0}{l} \left(\prod_{j=1}^{m} \binom{|\delta_{n_{i_1 + i_2 + \cdots + i_j}}(V)|}{i_j} \right) \left(\prod_{i=1}^{k} f((n_{\pi(i)} \vee n_{\pi(i+1)})\, \Delta) \right),$$

where the sum runs over all $n_1, \dots, n_k$ such that

$$n_k = \cdots = n_{k-l+1} = 0 < n_{i_1 + i_2 + \cdots + i_m} = \cdots = n_{i_1 + i_2 + \cdots + i_{m-1} + 1}$$
$$< \cdots < n_{i_1 + i_2} = \cdots = n_{i_1 + 1} < n_{i_1} = \cdots = n_1 \leqslant t.$$

Substituting (4.9) and (4.10) into (4.20), and using (4.19), we derive

$$\begin{aligned}
\mathbf{E}_{\bar V} C_k^{(S)}(S+T) &= \frac{1}{k}\,\lambda_2^k(\theta(T)+\varepsilon(\Delta))^k\,(1+\varepsilon(\Delta^{1/2}T))(1+\varepsilon(\bar V_0^{-1}))\\
&\quad+\frac{1}{k}\sum_{l=0}^{k-1}\sum_{\pi\in\mathscr{P}_k}\Bigg\{(1+\varepsilon(\Delta^{1/2}))(1+\varepsilon(\bar V_0^{-1}))\,\lambda_1^{k-l}\,\frac{1}{\bar V_0^l}\binom{\bar V_0}{l}\\
&\quad\times\sum_{n_k=\cdots=n_{k-l+1}=0<n_{k-l}<\cdots<n_1\leqslant t}\Delta^{k-l}\lambda_2^k(1+\varepsilon(\Delta^{1/2}T))\\
&\quad\times\left(\prod_{i=1}^{k-l}e^{\Delta\lambda_1 n_i}\right)\prod_{i=1}^{k}\exp\{-\Delta\lambda_1(n_{\pi(i)}\vee n_{\pi(i+1)})\}\\
&\quad\times(\theta(T-(n_{\pi(i)}\vee n_{\pi(i+1)})\,\Delta)+\varepsilon(\Delta))+D_k(\pi,l)\Bigg\}. \qquad (4.22)
\end{aligned}$$

Due to the uniform in T (recall that $T>1$ by the assumption) boundedness of the following below integrals (4.24), the last formula yields

$$\begin{aligned}
\mathbf{E}_{\bar V} C_k^{(S)}(S+T) &= \frac{1}{k}\,\lambda_2^k(\theta(T)+\varepsilon(\Delta))^k\,(1+\varepsilon(\Delta^{1/2}T))(1+\varepsilon(\bar V_0^{-1}))\\
&\quad+\frac{1}{k}\sum_{l=0}^{k-1}\sum_{\pi\in\mathscr{P}_k}\Bigg\{(1+\varepsilon(\Delta^{1/2}T))(1+\varepsilon(\bar V_0^{-1}))\,\lambda_2^k\lambda_1^{k-l}\\
&\quad\times\frac{1}{l!}\,I_{kl}(\pi,T)+D_k(\pi,l)\Bigg\}+\varepsilon_0(\Delta), \qquad (4.23)
\end{aligned}$$

where $\varepsilon_0(\Delta)\to 0$ as $\Delta\to 0$, and for every $0\leqslant l\leqslant k-1$

$$\begin{aligned}
I_{kl}(\pi,T) &= \int\cdots\int_{0\leqslant s_{k-l}\leqslant\cdots\leqslant s_1\leqslant T}\left(\prod_{i=1}^{k}e^{\lambda_1 s_i}\right)\prod_{i=1}^{k}\exp\{-\lambda_1(s_{\pi(i)}\vee s_{\pi(i+1)})\}\\
&\quad\times(\theta(T-(s_{\pi(i)}\vee s_{\pi(i+1)}))+\varepsilon(\Delta))\,ds_{k-l}\cdots ds_1, \qquad (4.24)
\end{aligned}$$

with $s_{k-l+1}=\cdots=s_k=0$ unless $l=0$. First we shall show that the integrals $I_{kl}(\pi,T)$ for every $0\leqslant l\leqslant k-1$ are indeed bounded uniformly in $T>1$. With a change of variables rewrite integral $I_{kl}(\pi,T)$ as

$$\begin{aligned}
I_{kl}(\pi,T) &= \int\cdots\int_{0\leqslant s_1\leqslant\cdots\leqslant s_{k-l}\leqslant T}\left(\prod_{i=1}^{k}e^{-\lambda_1 s_i}\right)\\
&\quad\times\left(\prod_{i=1}^{k}\exp\{\lambda_1(s_{\pi(i)}\wedge s_{\pi(i+1)})\}(\theta(s_{\pi(i)}\wedge s_{\pi(i+1)})+\varepsilon(\Delta))\right)ds_1\cdots ds_{k-l}, \qquad (4.25)
\end{aligned}$$

where $s_{k-l+1} = \cdots = s_k = T$ unless $l = 0$. Notice that

$$e^{\lambda_1 s}\,\theta(s) = \frac{e^{(\lambda_1-\mu)\,s} - 1}{\lambda_1 - \mu}, \tag{4.26}$$

is an increasing function in s. Recall also a bound $|\varepsilon(\Delta)| \leqslant a\Delta$ where the constant $a > 0$ is independent of T (and of i, l and π). Thus by (4.25) we have uniformly in π

$$\begin{aligned} I_{k\,0}(\pi, T) &\leqslant \int_0^T \int_0^{s_k} \cdots \int_0^{s_2} \left(\prod_{i=1}^k e^{-\lambda_1 s_i}\right)\left(\prod_{i=1}^k e^{\lambda_1 s_i}\,(\theta(s_i) + a\Delta)\right) ds_1 \cdots ds_k \\ &< \frac{1}{k!}\left(\int_0^\infty \theta(s)\,ds + a\Delta T\right)^k \\ &= \frac{1}{k!}\left(\frac{1}{\lambda_1\,\mu} + a\Delta T\right)^k, \end{aligned} \tag{4.27}$$

which under the scaling (4.2) implies a uniform bound in T

$$I_{k\,0}(\pi, T) \leqslant \frac{1}{k!}\left(\frac{1}{\lambda_1\,\mu} + a\right)^k. \tag{4.28}$$

Also, it is easy to check producing similar to (4.27) bounds that we have under the scaling (4.2)

$$\begin{aligned} I_{k\,0}(\pi, T) = \int_0^T \int_0^{s_k} \cdots \int_0^{s_2} \left(\prod_{i=1}^k e^{-\lambda_1 s_i}\right) \\ \times \left(\prod_{i=1}^k \exp\{\lambda_1 (s_{\pi(i)} \wedge s_{\pi(i+1)})\}\,\theta(s_{\pi(i)} \wedge s_{\pi(i+1)})\right) ds_1 \cdots ds_k + \varepsilon(\Delta T). \end{aligned} \tag{4.29}$$

Next for every $1 \leqslant l \leqslant k-1$ we derive from (4.25) using again (4.26)

$$\begin{aligned} I_{k\,l}(\pi, T) &\leqslant (\theta(T) + a\Delta)^{l-1}\, e^{-\lambda_1 T} \int \cdots \int_{0 \leqslant s_1 \leqslant \cdots \leqslant s_{k-l} \leqslant T} \left(\prod_{i=1}^{k-l} e^{-\lambda_1 s_i}\right) \\ &\quad \times e^{\lambda_1 s_1}(\theta(s_1) + a\Delta)\left(\prod_{i=1}^{k-l} e^{\lambda_1 s_i}(\theta(s_i) + a\Delta)\right) ds_1 \cdots ds_{k-l} \\ &=: (\theta(T) + a\Delta)^{l-1}\, e^{-\lambda_1 T}\, \tilde{I}_{k-l}. \end{aligned} \tag{4.30}$$

Consider separately

$$e^{-\lambda_1 T}\tilde{I}_n \leqslant \frac{e^{-\lambda_1 T}}{(n-1)!}\int_0^T e^{\lambda_1 s_1}(\theta(s_1)+a\Delta)^2\left(\int_0^T (\theta(s)+a\Delta)\,ds\right)^{n-1} ds_1$$
$$< \frac{\max_{t>0}\theta(t)+a\Delta}{(n-1)!}\left(\frac{1}{\lambda_1\mu}+a\Delta T\right)^{n-1}$$
$$\times\left(a\Delta T+e^{-\lambda_1 T}\int_0^T e^{\lambda_1 y}\theta(y)\,dy\right). \tag{4.31}$$

Trivial computation yields under the scaling (4.2) a bound

$$e^{-\lambda_1 T}\tilde{I}_n \leqslant C(\Delta T+e^{-\min\{\lambda_1/2,\,\mu\}\,T}), \qquad 1\leqslant n\leqslant k-1,$$

for some positive constant C independent of T, which together with (4.30) yields under the scaling (4.2):

$$I_{k\,l}(\pi, T)\leqslant C_1^k(\Delta T+e^{-\min\{\lambda_1/2,\,\mu\}\,T}), \qquad 1\leqslant l\leqslant k-1 \tag{4.32}$$

for some positive constant C_1 independent of T (or S) and π. Similarly we obtain for all large enough $\bar{V}_0$ and small Δ:

$$D_k(\pi, l, m, \bar{\imath})\leqslant B_1^k\lambda_2^k\Delta^{k-l-m}\int\cdots\int_{0\leqslant s_1\leqslant\cdots\leqslant s_m\leqslant T}\left(\prod_{j=1}^m e^{-\lambda_1 i_j s_j}\right)$$
$$\times\left(\prod_{j=1}^m e^{\lambda_1 i_j s_j}\,(\theta(s_j)+a\Delta)^{i_j}\right)ds_1\cdots ds_m$$
$$< B_1^k\lambda_2^k\Delta^{k-l-m}\prod_{j=1}^m\int_0^T (\theta(s)+a\Delta)^{i_j}\,ds$$
$$\leqslant A_1^k\lambda_2^k\Delta^{k-l-m}, \tag{4.33}$$

where B_1 and A_1 are some constants depending on λ_1 and μ only. Substituting this bound into the definition (4.21) we get uniformly in π and $0\leqslant l\leqslant k-2$

$$D_k(\pi, l)\leqslant A_3^k\Delta \tag{4.34}$$

for some constant $A_3=A_3(\lambda_1,\mu)$ independent of Δ. Now armed with bounds (4.32) and (4.34) we are able to derive from (4.23) under the scaling (4.2)

$$\mathbf{E}_{\bar{V}}C_k^{(S)}(S+T)=\frac{1}{k}\sum_{\pi\in\mathscr{P}_k}(1+\varepsilon(\bar{V}_0^{-1}))(1+\varepsilon(\Delta^{1/2}T))$$
$$\times(\lambda_1\lambda_2)^k\,I_{k0}(\pi, T)+\varepsilon_0(\Delta)+\varepsilon(\Delta). \tag{4.35}$$

It follows from (4.35) and the bound (4.28), that $\mathbf{E}_{\bar{V}} C_k^{(S)}(S+S^{3/2})$ is uniformly bounded in $S>1$ and in $\bar{V} \in \mathscr{A}(S)$. Thus with a help of (3.13) we get

$$\lim_{S\to\infty} \mathbf{E} C_k^{(S)}(S+S^{3/2})\, \mathbf{1}_{\mathscr{A}(S)} \mathbf{1}_{\{|V(S)| \leqslant S\}} = 0. \tag{4.36}$$

Now under the assumption (4.2) we easily derive from (4.35) and (4.36) taking into account (4.29)

$$\begin{aligned}
&\lim_{S\to\infty} \mathbf{E} C_k^{(S)}(S+S^{3/2})\, \mathbf{1}_{\mathscr{A}(S)} \\
&\quad = \lim_{S\to\infty} \mathbf{E} C_k^{(S)}(S+S^{3/2})\, \mathbf{1}_{\mathscr{A}(S)} \mathbf{1}_{\{|V(S)| > S\}} \\
&\quad = \lim_{S\to\infty} \frac{1}{k} (\lambda_1 \lambda_2)^k \sum_{\pi \in \mathscr{P}_k} I_{k\,0}(\pi, S^{3/2}) \\
&\quad = \frac{1}{k} (\lambda_1 \lambda_2)^k \sum_{\pi \in \mathscr{P}_k} \int_0^\infty \int_0^{s_k} \cdots \int_0^{s_2} \left(\prod_{i=1}^k e^{-\lambda_1 s_i} \right) \\
&\qquad \times \left(\prod_{i=1}^k \exp\{\lambda_1 (s_{\pi(i)} \wedge s_{\pi(i+1)})\}\, \theta(s_{\pi(i)} \wedge s_{\pi(i+1)}) \right) ds_1 \cdots ds_k
\end{aligned}$$

for any fixed finite k. This implies (4.5) and finishes the proof of Lemma 4.1. ∎

Clearly, this result and (4.4) will yield the statement of Theorem 1.2 as soon as we show that the last two limits in (4.4) exist and equal zero. This will be the subject of two following lemmas.

Lemma 4.2. For any finite $k \geqslant 3$

$$\lim_{S\to\infty} \mathbf{E} C_k(S+S^{3/2})\, \mathbf{1}_{\Omega \setminus \mathscr{A}(S)} = 0. \tag{4.37}$$

Proof. In the following we use again the notations (4.2). Let a trajectory $\bar{V} = \{|V(t)| = \bar{V}_t, 0 < t \leqslant S+T\}$ be fixed arbitrarily but so that $\mathbf{P}\{\bar{V}\} > 0$. Conditionally on $\bar{V}$ let us denote $p_{\bar{V}}(u_s, v_\tau)$ the probability of an edge in the graph $\mathscr{G}^d(S+T)$ from a vertex $u_s \in V(s) \setminus V(s-\Delta)$ to a vertex $v_\tau \in V(\tau) \setminus V(\tau-\Delta)$, where $0 < s, \tau < S+T$. Then according to (2.16) and (2.4) we have the following bound:

$$p_{\bar{V}}(u_s, v_\tau) \leqslant A_1 \left(\frac{\Delta}{\bar{V}_{\tau\vee s}} + \sum_{n=\left[\frac{\tau\vee s}{\Delta}\right]+1}^{\left[\frac{S+T}{\Delta}\right]} \Delta e^{-\mu(S+T-\Delta n)} \frac{1}{\bar{V}_{\Delta(n-1)}} \right) =: g(\tau \vee s) \tag{4.38}$$

for some constant A_1 independent of s, τ and S. Now similar to (4.17) we obtain

$$\mathbf{E}_{\bar{V}} C_k(S+T) \leqslant \frac{1}{k} \sum_{0<n_k \leqslant \cdots \leqslant n_1 \leqslant \left[\frac{S+T}{\Delta}\right]} \sum_{\{v^1,\dots,v^k\}:\, v^i \in V(n_i\Delta)\setminus V((n_i-1)\Delta)} \times \sum_{\pi \in \mathscr{P}_k} \prod_{i=1}^{k} g((n_{\pi(i)} \vee n_{\pi(i+1)})\,\Delta),$$

where conditionally on $\bar{V}$ we treat $V(n_i\Delta)\setminus V((n_i-1)\,\Delta)$ simply as a set of $\bar{V}_{n_i\Delta} - \bar{V}_{(n_i-1)\,\Delta}$ different elements as we have done earlier. Continuing this bound we derive taking into account that $g(s)$ is a decreasing function

$$\mathbf{E}_{\bar{V}} C_k(S+T) \leqslant (k-1)!\, A_2^k \sum_{m=1}^{k} \sum_{0<n_m<\cdots<n_1 \leqslant \left[\frac{S+T}{\Delta}\right]} \times \sum_{(i_1,\dots,i_m):\, i_1+\cdots+i_m=k,\, i_j \geqslant 1} \prod_{j=1}^{m} ((\bar{V}_{n_j\Delta} - \bar{V}_{(n_j-1)\,\Delta})\, g(n_j\Delta))^{i_j}, \tag{4.39}$$

for some constant A_2 independent of S. Since the function

$$(\bar{V}_{n_j\Delta} - \bar{V}_{(n_j-1)\,\Delta})\, g(n_j\Delta) = A_1\Delta\left(1 - \frac{\bar{V}_{(n_j-1)\,\Delta}}{\bar{V}_{n_j\Delta}}\right) + A_1 \sum_{n=n_j+1}^{\left[\frac{S+T}{\Delta}\right]} \Delta e^{-\mu(S+T-\Delta n)} \frac{(\bar{V}_{n_j\Delta} - \bar{V}_{(n_j-1)\,\Delta})}{\bar{V}_{\Delta(n-1)}} \tag{4.40}$$

is positive and bounded uniformly in $\bar{V}$, $0 \leqslant \Delta < 1$ and $S > 1$, we derive from (4.39)

$$\begin{aligned}
\mathbf{E}_{\bar{V}} C_k(S+T) &\leqslant A_3 \sum_{m=1}^{k} \sum_{0<n_m<\cdots<n_1 \leqslant \left[\frac{S+T}{\Delta}\right]} \times \sum_{(i_1,\dots,i_m):\, i_1+\cdots+i_m=k,\, i_j \geqslant 1} \prod_{j=1}^{m} ((\bar{V}_{n_j\Delta} - \bar{V}_{(n_j-1)\,\Delta})\, g(n_j\Delta)) \\
&\leqslant A_4 \sum_{m=1}^{k} \prod_{j=1}^{m} \sum_{n_j=0}^{\left[\frac{S+T}{\Delta}\right]} (\bar{V}_{n_j\Delta} - \bar{V}_{(n_j-1)\,\Delta})\, g(n_j\Delta) \\
&= A_4 \sum_{m=1}^{k} \left(\sum_{l=0}^{\left[\frac{S+T}{\Delta}\right]} (\bar{V}_{l\Delta} - \bar{V}_{(l-1)\,\Delta})\, g(l\Delta) \right)^m,
\end{aligned}$$

where the positive coefficients A_3 and A_4 are independent of S (but depend on k and the parameters of the model). Recalling formula (4.40) we get now with a help of Lyapunov's inequality

$$\begin{aligned}
\mathbf{E}_{\bar{V}} C_k(S+T) &\leqslant A_5 \sum_{m=1}^{k} \left(\Delta \sum_{n=0}^{\left[\frac{S+T}{\Delta}\right]} \left(1 - \frac{\bar{V}_{(n-1)\Delta}}{\bar{V}_{n\Delta}} \right) \right)^m \\
&\quad + A_5 \sum_{m=1}^{k} \left(\sum_{n=1}^{\left[\frac{S+T}{\Delta}\right]} \Delta \frac{e^{-\mu(S+T-\Delta n)}}{\bar{V}_{\Delta(n-1)}} \sum_{l=0}^{n-1} (\bar{V}_{l\Delta} - \bar{V}_{(l-1)\Delta}) \right)^m \\
&\leqslant A_6 \sum_{m=1}^{k} \Delta^m \left(\frac{S+T}{\Delta} \right)^{m-1} \sum_{n=0}^{\left[\frac{S+T}{\Delta}\right]} \left(1 - \frac{\bar{V}_{(n-1)\Delta}}{\bar{V}_{n\Delta}} \right)^m + A_6,
\end{aligned} \tag{4.41}$$

where the positive coefficients A_5 and A_6 are independent of S. This bound implies

$$\begin{aligned}
&\mathbf{E} C_k(S+T)\, \mathbf{1}_{\Omega \setminus \mathscr{A}(S)} \\
&\quad \leqslant A_6 \sum_{m=1}^{k} \Delta\, (S+T)^{m-1} \sum_{n=0}^{\left[\frac{S+T}{\Delta}\right]} \mathbf{E} \left(1 - \frac{|V((n-1)\Delta)|}{|V(n\Delta)|} \right)^m \\
&\qquad + A_6 (1 - \mathbf{P}\{\mathscr{A}(S)\}).
\end{aligned} \tag{4.42}$$

Using formula (3.13) it is easy to derive for any $1 \leqslant m \leqslant k$ and $n \geqslant 0$

$$\mathbf{E} \left(1 - \frac{|V((n-1)\Delta)|}{|V(n\Delta)|} \right)^m \leqslant A_7 \Delta$$

for some positive constant $A_7 = A_7(k, \lambda_1)$ independent of S and $n \geqslant 0$. Substituting this bound into (4.42) we obtain

$$\mathbf{E} C_k(S+T)\, \mathbf{1}_{\Omega \setminus \mathscr{A}(S)} \leqslant A_6 A_7 \sum_{m=1}^{k} \Delta (S + S^{3/2} + 1)^m + A_6 (1 - \mathbf{P}\{\mathscr{A}(S)\}), \tag{4.43}$$

where $\Delta = S^{-3(k+2)/2}$ according to (4.2). Thus the last bound together with (4.3) immediately yields the statement (4.37). ▮

Lemma 4.3. For any finite $k \geqslant 3$

$$\lim_{S \to \infty} \mathbf{E} C_k^{(<S)}(S + S^{3/2})\, \mathbf{1}_{\mathscr{A}(S)} = 0. \tag{4.44}$$

Proof. Note that in the graph $\mathcal{G}^d(S+T)$ the only vertices between which there might be an edge of age greater or equal T are the vertices in the set $V(S)$. Define for every $2 \leqslant l \leqslant k-1$ and $\pi \in \mathcal{P}_k$ with $\pi(k+1) \equiv \pi(1)$ a set

$$\mathcal{M}_l(\pi) = \{1 \leqslant i \leqslant k : \{\pi(i), \pi(i+1)\} \in \{k-l+1,..., k\}\}.$$

Let $\bar{V} \in \mathcal{A}(S)$ be fixed arbitrarily. Obviously, we have a bound

$$\begin{aligned}
\mathbf{E}_{\bar{V}} C_k^{(<S)}(S+T) \leqslant & \frac{1}{k} \sum_{(v^1,..., v^k) : v^i \in \delta_0(V)} \mathbf{P}_{\bar{V}} \{C(v^1,..., v^k) \in \mathcal{G}^d(S+T)\} \\
& + \frac{1}{k} \sum_{l=2}^{k-1} \sum_{0 = n_k = \cdots = n_{k-l+1} < n_{k-l} \leqslant \cdots \leqslant n_1 \leqslant t} \\
& \times \sum_{\{v^1,..., v^k\} : v^i \in \delta_{n_i}(V)} \sum_{\pi \in \mathcal{P}_k : \mathcal{M}_l(\pi) \neq \varnothing} \mathbf{P}_{\bar{V}} \{C(\pi(v)) \in \mathcal{G}^d(S+T)\},
\end{aligned} \tag{4.45}$$

where $t = [\frac{T}{\Delta}]$, $(v^1,..., v^k)$ denotes a vector, and $\{v^1,..., v^k\}$ denotes a set. Then similar to (4.20) and (4.22) we derive using formula (4.9) and bound (4.13):

$$\begin{aligned}
\mathbf{E}_{\bar{V}} C_k^{(<S)}(S+T) \leqslant & B_1 \bar{V}_0^k \left(\frac{\theta(T) + a\Delta}{\bar{V}_0} + Ae^{-\mu T} \right)^k \\
& + B_1 \sum_{l=2}^{k-1} \sum_{\pi \in \mathcal{P}_k : \mathcal{M}_l(\pi) \neq \varnothing} \bar{V}_0^l \left(\frac{\theta(T) + a\Delta}{\bar{V}_0} + Ae^{-\mu T} \right)^{|\mathcal{M}_l(\pi)|} \\
& \times \sum_{0 < n_{k-l} \leqslant \cdots \leqslant n_1 \leqslant t} \sum_{\{v^{k-l},..., v^k\} : v^i \in \delta_{n_i}(V)} \\
& \times \left(\prod_{i \notin \mathcal{M}_l(\pi)} f((n_{\pi(i)} \vee n_{\pi(i+1)}) \Delta) \right),
\end{aligned} \tag{4.46}$$

where for each lth term $0 = n_k = \cdots = n_{k-l+1}$, and the positive constants a and B_1 are independent of T and S. We write here $i \notin \mathcal{M}_l(\pi)$ meaning $i \in \{1,..., k\} \setminus \mathcal{M}_l(\pi)$. Now for each $2 \leqslant l \leqslant k-1$ we shall bound from above the sum of the terms in (4.46) which contain no ties among $n_{k-l},..., n_1$. Thus for any $2 \leqslant l \leqslant k-1$ and $\mathcal{M}_l(\pi) \neq \varnothing$ we obtain using (4.18) and (4.19), as well as taking into account the uniform boundedness of the following below integral (4.48)

$$\bar{V}_0^{\,l-|\mathcal{M}_l(\pi)|} \sum_{0<n_{k-l}<\cdots<n_1\leqslant t} \sum_{\{v^{k-l},\ldots,v^k\}:\, v^i\in\delta_{n_i}(V)} \prod_{i\notin\mathcal{M}_l(\pi)} f((n_{\pi(i)}\vee n_{\pi(i+1)})\,\Delta)$$
$$\leqslant B_2\bar{V}_0^{\,l-|\mathcal{M}_l(\pi)|} \sum_{0<n_{k-l}<\cdots<n_1\leqslant t} \left(\prod_{i=1}^{k-l} \Delta\bar{V}_{n_i\Delta}\right) \prod_{i\notin\mathcal{M}_l(\pi)} f((n_{\pi(i)}\vee n_{\pi(i+1)})\,\Delta)$$
$$\leqslant B_2 J_T(k,l,\pi)+B_2', \tag{4.47}$$

for all large $\bar{V}_0$ and small Δ, where B_2 and B_2' are some positive constants independent of T and S, and

$$J_T(k,l,\pi)=\int\cdots\int_{0\leqslant s_{k-l}\leqslant\cdots\leqslant s_1\leqslant T}\left(\prod_{i=1}^{k-l} e^{\lambda_1 s_i}\right)$$
$$\times\left(\prod_{i\notin\mathcal{M}_l(\pi)} \exp\{-\lambda_1(s_{\pi(i)}\vee s_{\pi(i+1)})\}\right.$$
$$\left.\times\,(\theta(T-(s_{\pi(i)}\vee s_{\pi(i+1)}))+a\Delta)\right) ds_{k-l}\cdots ds_1,$$

with $s_{k-l+1}=\cdots=s_k=0$. With a change of variables rewrite this integral as

$$J_T(k,l,\pi)=\int\cdots\int_{0\leqslant s_1\leqslant\cdots\leqslant s_{k-l}\leqslant T} e^{-\lambda_1(l-|\mathcal{M}_l(\pi)|)\,T}\left(\prod_{i=1}^{k-l} e^{-\lambda_1 s_i}\right)$$
$$\times\left(\prod_{i\notin\mathcal{M}_l(\pi)} \exp\{\lambda_1(s_{\pi(i)}\wedge s_{\pi(i+1)})\}\right.$$
$$\left.\times\,(\theta(s_{\pi(i)}\wedge s_{\pi(i+1)})+a\Delta)\right) ds_1\cdots ds_{k-l}, \tag{4.48}$$

where $s_{k-l+1}=\cdots=s_k=T$. Trivial, but worth noticing that $|\mathcal{M}_l(\pi)|\leqslant l$ for all $l\leqslant k-1$ and π. Using (4.26) we derive from here the following bound

$$J_T(k,l,\pi)\leqslant(\max_{0\leqslant s\leqslant T}\theta(s)+a\Delta)^{l-|\mathcal{M}_l(\pi)|}$$
$$\times\int\cdots\int_{0\leqslant s_1\leqslant\cdots\leqslant s_{k-l}\leqslant T}\left(\prod_{i=1}^{k-l}(\theta(s_i)+a\Delta)\right) ds_1\cdots ds_{k-l}$$
$$<(\max_{0\leqslant s\leqslant T}\theta(s)+a\Delta)^{l-|\mathcal{M}_l(\pi)|}\,\frac{1}{(k-l)!}\left(\frac{1}{\lambda_1\mu}+a\Delta T\right)^{k-l},$$

which due to the scaling (4.2) and boundedness of the functions θ yields

$$J_T(k, l, \pi) < B_3, \tag{4.49}$$

where B_3 is some positive constant independent of S. With a similar argument it is not difficult to derive (as we did to get (4.34)) a bound for the rest of the terms in (4.46) for all $2 \leqslant l \leqslant k-2$:

$$\begin{aligned}&\bar{V}_0^{l-|\mathcal{M}_l(\pi)|} \sum_{0<n_{k-l} \leqslant \cdots \leqslant n_1 \leqslant t:\, n_i = n_j \text{ for some } 1 \leqslant i < j \leqslant k-l} \\ &\times \sum_{\{v^{k-l},\ldots,v^k\}:\, v^i \in \delta_{n_i}(V)} \Bigg(\prod_{i \notin \mathcal{M}_l(\pi)} f((n_{\pi(i)} \vee n_{\pi(i+1)})\, \Delta) \Bigg) \leqslant B_4 \Delta,\end{aligned} \tag{4.50}$$

for some constant B_4 independent of S and T. Combining (4.50), (4.49) and (4.47) with (4.46) we readily derive taking into account the scaling (4.2)

$$\mathbf{E}_{\bar{V}} C_k^{(<S)}(S+T) \leqslant B_5 \sum_{m=1}^{k} (\theta(T) + a\Delta + A\bar{V}_0 e^{-\mu T})^m, \tag{4.51}$$

where B_5 is some positive constant independent of S. This bound together with the scaling (4.2) yields

$$\mathbf{E} C_k^{(<S)}(S+S^{3/2}) \leqslant B_6 \sum_{m=1}^{k} (\theta^m(S^{3/2}) + S^{-m3(k+2)/2} + e^{-\mu m S^{3/2}} \mathbf{E}\, |V(S)|^m) \tag{4.52}$$

where positive constant B_6 is independent of S. Using formula (3.13) it is easy to get for any $1 \leqslant m \leqslant k$

$$\mathbf{E}\, |V(S)|^m < B_7 e^{2\lambda_1 m S}$$

for some constant $B_7 = B_7(k, \lambda_1)$ independent of S. Substituting this bound into (4.52) and passing to the limit as $S \to \infty$, we immediately obtain the statement of Lemma 4.3. ∎

The results of the last two lemmas together with (4.5) and (4.4) complete the proof of Theorem 1.2. ∎

Remark 1.4 follows by Lemma 3.1.

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REFERENCES

1. S. H. Strogatz, Exploring complex networks, *Nature* **410**:268–276 (2001).
2. V. A. Malyshev, Random graphs and grammars on graphs, *Discrete Math. Appl.* **8**: 247–262 (1998).
3. D. J. Watts and S. H. Strogatz, Collective dynamics of "small-world" networks, *Nature* **393**:440–442 (1998).
4. A. D. Barbour and G. Reinert, Small worlds, *Random Structures Algorithms* **19**:54–74 (2001).
5. A.-L. Barabási, R. Albert, and H. Jeong, Mean-field theory for scale-free random networks, *Phys. A* **272**:173–187 (1999).
6. B. Bollobás, O. Riordan, J. Spencer, and G. Tusnády, The degree sequence of a scale-free random graph process, *Random Structures Algorithms* **18**:279–290 (2001).
7. D. S. Callaway, J. E. Hopcroft, J. M. Kleinberg, M. E. J. Newman, and S. H. Strogatz, Are randomly grown graphs really random? *Phys. Rev. E* **64**:041902 (2001).
8. P. Erdös, Some remarks on the theory of graphs, *Bull. Amer. Math. Soc.* **53**:292–294 (1947).
9. B. Bollobás, *Random Graphs* (Academic Press, 1985).
10. S. Janson, T. Łuczak, and A. Ruciński, *Random Graphs* (Wiley-Interscience, New York, 2000).
11. T. S. Turova, Dynamical random graphs with memory, *Phys. Rev. E* **65**:066102 (2002).
12. K. B. Athreya and P. E. Ney, *Branching Processes*, Die Grundlehren der mathematischen Wissenschaften, Band 196 (Springer-Verlag, New York/Heidelberg, 1972).
13. E. M. Jin, M. Girvan, and M. E. J. Newman, Structure of growing social networks, *Phys. Rev. E* **64**:046132 (2001).
14. J. Xing and G. L. Gerstein, Networks with lateral connectivity I, II, III, *J. Neurophysiology* **75**:184–231 (1996).
15. Y. Prut, E. Vaadia, H. Bergman, I. Haalman, H. Slovin, and M. Abeles, Spatiotemporal structure of cortical activity: Properties and behavioral relevance, *J. Neurophysiology* **79**: 2857–2874 (1998).
16. S. N. Dorogovtsev and J. F. F. Mendes, Scaling properties of scale-free evolving networks: continuous approach, *Phys. Rev. E* **63**:056125 (2001).
17. R. Albert, H. Jeong, and A.-L. Barabási, Error and attack tolerance of complex networks, *Nature* **406**:378–382 (2000).
18. T. S. Turova, Neural networks through the hourglass, *BioSystems* **58**:159–165 (2000).
19. J. Jonasson, The random triangle model, *J. Appl. Probab.* **36**:852–867 (1999).
20. O. Häggström and T. S. Turova, A strict inequality for the random triangle model, *J. Statist. Phys.* **104**:471–482 (2001).
21. M. Krikun, Height of a random tree, *Markov Process. Related Fields* **6**:135–146 (2000).
22. Y. Peres, Probability on trees: An introductory climb, *Lectures on Probability Theory and Statistics* (Saint-Flour, 1997), *Lecture Notes in Math.* **1717**:193–280 (Springer, Berlin, 1999).
23. T. S. Turova, Long paths and cycles in the dynamical graphs, preprints in *Math. Sciences* **2001**:13 (2001).